

H -CONTACT UNIT TANGENT SPHERE BUNDLES OF RIEMANNIAN MANIFOLDS

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ABSTRACT. A contact metric manifold is said to be H -contact, if the characteristic vector field is harmonic. We prove that the unit tangent bundle of a Riemannian manifold M equipped with the standard contact metric structure is H -contact if and only if M is 2-stein.

1. INTRODUCTION

Let (\tilde{M}, \tilde{g}) be a compact, orientable Riemannian manifold. The *energy* $E(V)$ of a unit vector field V is defined as the energy of the corresponding map between (\tilde{M}, \tilde{g}) and its tangent sphere bundle equipped with the Sasaki metric:

$$E(V) = \frac{1}{2} \int_M |dV|^2 dv_{\tilde{g}} = \frac{m}{2} \text{Vol}(\tilde{M}, \tilde{g}) + \frac{1}{2} \int_M |\nabla V|^2 dv_{\tilde{g}},$$

where $m = \dim M$ [16]. Wood defined a unit vector field to be *harmonic* if it is a critical point for the energy functional E in the set of all unit vector fields of \tilde{M} , and then by considering the first variation obtained a local condition for harmonicity of a vector field [16].

A contact metric manifold whose characteristic vector field ξ is harmonic is called an H -contact manifold. Perrone proved that a contact metric manifold is H -contact if and only if the characteristic vector field ξ is an eigenvector of the Ricci operator [14].

A substantial progress has been achieved in the study of this construction in the case when the contact metric manifold is the unit tangent sphere bundle of a Riemannian manifold (M, g) equipped with the Sasaki metric and the standard contact structure. Boeckx and Vanhecke [5] showed that the unit tangent sphere bundle of a 2-dimensional or a 3-dimensional Riemannian manifold is H -contact if and only if the base manifold (M, g) has constant sectional curvature. In [7], Calvaruso and Perrone proved that the same is true under assumption that (M, g) is conformally flat, and also obtained a local characterisation of such manifolds (M, g) in the general case (see Proposition 1 in Section 3 below). That characterisation was used in [10] to show that a Riemannian manifold whose unit tangent sphere bundle is H -contact has constant scalar curvature, constant norm of the Ricci tensor and constant norm of the curvature tensor (the latter is true when $\dim M \neq 4$; there is a counterexample in dimension 4). It was further established that a Riemannian manifold whose unit tangent sphere bundle is H -contact is 2-stein, provided that either (M, g) is Einstein [9] or $\dim M = 4$ [11].

Our main result is as follows.

Theorem. *Let (M, g) be a Riemannian manifold. The unit tangent sphere bundle T_1M equipped with the standard contact metric structure is H -contact if and only if (M, g) is 2-stein.*

Recall that an n -dimensional Riemannian manifold (M, g) is said to be *2-stein* if there exist two functions $f_1, f_2 : M \rightarrow \mathbb{R}$ such that for every $p \in M$ and every vector X tangent to M at p we have

$$\text{Tr } R_X = f_1(p) \|X\|^2, \quad \text{Tr}(R_X^2) = f_2(p) \|X\|^4,$$

where R_X is the Jacobi operator at p [8, p. 47]. In particular, any 2-stein manifold is Einstein. By Schur's Theorem, the function f_1 is constant when $n \geq 3$, and by [1, §6.57, 6.61], the function f_2 is constant when $n \geq 5$. 2-stein manifolds show up in many questions in Riemannian geometry, to name one, in the theory of harmonic

2010 *Mathematics Subject Classification.* Primary 53C25, 53D10; Secondary 53B20.

Key words and phrases. tangent sphere bundle, H -contact manifold, 2-stein.

Y. Nikolayevsky was supported by ARC Discovery grant DP130103485 and by Korea Institute for Advanced Study.

J.H. Park was supported by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2014053413).

spaces [1, §6.46]. In dimension $n = 4$, the structure of the curvature tensor of a 2-stein manifold is well known [15, Lemma 7]; any four-dimensional 2-stein manifold is pointwise Osserman [1, §6.73], hence self-dual (up to the choice of orientation) [3], and as such, is either hyperkähler or quaternionic Kähler depending on the scalar curvature. In dimension $n = 5$ one can be much more specific (perhaps the true reason for that is the fact that f_2 is constant): any 2-stein manifold either has constant curvature, or up to scaling, is locally isometric either to the symmetric space $SU(3)/SO(3)$ or to its non-compact dual $SL(3)/SO(3)$ [13, Proposition 1]. The classification of 2-stein spaces is known in the locally symmetric case [8] and for some other classes of manifolds.

In Section 2 we give necessary background on contact geometry and the Sasaki metric. The proof of the Theorem is given in Section 3; the core of the proof is a purely algebraic Proposition 2.

All the objects (manifolds, metrics, vector fields, forms) in this paper are assumed to be of class C^∞ .

2. STANDARD CONTACT METRIC STRUCTURE ON THE UNIT TANGENT SPHERE BUNDLE

We start with some preliminaries on a contact metric manifolds (the reader is referred to [2] for more details). A $(2n-1)$ -dimensional manifold \bar{M} is said to be *contact* if it admits a global 1-form η such that $\eta \wedge (d\eta)^{n-1} \neq 0$ everywhere on \bar{M} , where the exponent denotes the $(n-1)$ -st exterior power. We call such an η a *contact form* on \bar{M} . Given a contact form η , there exists a unique vector field ξ , the *characteristic vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$, for any vector field X on \bar{M} . A Riemannian metric \bar{g} on \bar{M} is said to be an *associated metric* to a contact form η if there exists a $(1, 1)$ -tensor field ϕ satisfying

$$\eta(X) = \bar{g}(X, \xi), \quad d\eta(X, Y) = \bar{g}(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \quad (1)$$

for any vector fields X and Y on \bar{M} . A Riemannian manifold \bar{M} equipped with structure tensors $(\bar{g}, \phi, \xi, \eta)$ satisfying (1) is called a *contact metric manifold*.

Let (M, g) be a Riemannian manifold, with the Levi-Civita connection ∇ . The tangent bundle TM of (M, g) consists of pairs (p, u) , where $p \in M$ and u a tangent vector to M at p . The mapping $\pi : TM \rightarrow M$, $\pi(p, u) = p$, is the natural projection from TM onto M .

For a vector field X on M , its *vertical lift* X^v is the unique vector field on TM defined by $X^v \omega = \omega(X) \circ \pi$, where ω is a 1-form on M , and the *horizontal lift* X^h is the unique vector field on TM defined by $X^h \omega = \nabla_X \omega$. Any vector tangent to TM at (p, u) can be uniquely represented as $X^h + Y^v$ for some vectors X and Y tangent to M at p . The tangent bundle TM can be endowed in a natural way with a Riemannian metric g_S , the *Sasaki metric* defined as follows:

$$g_S(X^h, Y^h) = g_S(X^v, Y^v) = g(X, Y) \circ \pi, \quad g_S(X^h, Y^v) = 0,$$

for any vector fields X and Y on M (for more details on the Sasaki metric we refer the reader to the survey papers [4, 6]). The Sasaki metric g_S is Hermitian for the almost complex structure J defined by $JX^h = X^v$ and $JX^v = -X^h$.

The unit tangent sphere bundle is the hypersurface of TM given by $g_p(u, u) = 1$. The unit normal vector field $N = u^v$ to $T_1 M$ is the vertical lift of u to $(p, u) \in T_1 M$.

We now define the standard contact metric structure of the unit tangent sphere bundle $T_1 M$ of a Riemannian manifold (M, g) . The metric g' on $T_1 M$ is induced from the Sasaki metric g_S on TM . Using the almost complex structure J on TM , we define the unit vector field ξ' , the 1-form η' and the $(1, 1)$ tensor field ϕ' on $T_1 M$ by $\xi' = -JN$ and $\phi' = J - \eta' \otimes N$. Since $g'(\bar{X}, \phi' \bar{Y}) = 2d\eta'(\bar{X}, \bar{Y})$, the quadruple (g', ϕ', ξ', η') is not a contact metric structure. By rescaling

$$\xi = 2\xi', \quad \eta = \frac{1}{2}\eta', \quad \phi = \phi', \quad \bar{g} = \frac{1}{4}g',$$

we get the standard contact metric structure $(\bar{g}, \phi, \xi, \eta)$ on $T_1 M$.

3. PROOF OF THE THEOREM

Let (M, g) be an n -dimensional Riemannian manifold, with ∇, R and ρ the Levi-Civita connection, the Riemann curvature tensor and the Ricci tensor respectively.

Suppose that the unit tangent sphere bundle $T_1 M$ equipped with the standard contact metric structure is H -contact.

By [5, Proposition 2] and [11] we can assume that $n \geq 5$. By [7, Proposition 3.1] we have the following.

Proposition 1. *The unit tangent sphere bundle T_1M of a Riemannian manifold (M, g) is H -contact with respect to the standard contact metric structure $(\bar{g}, \phi, \xi, \eta)$ if and only if the following two conditions are satisfied.*

(a) *The Ricci tensor is Codazzi, that is, for arbitrary vector fields X, Y and Z on M we have*

$$\nabla_X \rho(Y, Z) = \nabla_Y \rho(X, Z). \quad (2)$$

(b) *For unit, orthogonal vector fields X, Y on M , we have*

$$\sum_{i=1}^n g(R(X, e_i)X, R(X, e_i)Y) = 2\rho(X, Y), \quad (3)$$

where $\{e_i\}_{i=1}^n$ is a (local) orthonormal frame on M .

By the result of [9], it suffices to prove that (M, g) is Einstein. Seeking a contradiction assume that it is not. Let $p \in M$ be a point having the maximal number, N , of pairwise non-equal Ricci eigenvalues. By our assumption $N \geq 2$, and by construction, in a neighbourhood of p , the Ricci tensor ρ has N smooth eigendistributions, of constant dimensions. Let E_1, \dots, E_N be the eigenspaces of ρ at p . By [12, Theorem 1] the fact that ρ is a Codazzi tensor implies that for $1 \leq \lambda, \mu, \nu \leq N$,

$$R(E_\lambda, E_\mu)E_\nu = 0, \quad \text{when } \nu \notin \{\lambda, \mu\}. \quad (4)$$

We now consider condition (b). Given a Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ (with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$), we define an *algebraic curvature tensor* to be a $(4, 0)$ tensor having the same algebraic symmetries as the curvature tensor of a Riemannian manifold. Given an algebraic curvature tensor \mathcal{R} , we can define the corresponding Ricci tensor, and for any $X \in \mathbb{R}^n$, the Jacobi operator \mathcal{R}_X .

Take $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ to be the tangent space to M at the point p and define the algebraic curvature tensor \mathcal{R} on $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ by

$$\mathcal{R}(X, Y, Z, W) = R(X, Y, Z, W) - 2(\langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle). \quad (5)$$

Note that \mathcal{R} is obtained from R by shifting by an algebraic curvature tensor of constant curvature, and so equation (4) with R replaced by \mathcal{R} is still satisfied. What is more, we have the following fact.

Lemma 1. *Let R be an algebraic curvature tensor in $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. The following two conditions are equivalent.*

(a) *The algebraic curvature tensor R satisfies (3), for any unit, orthogonal vectors $X, Y \in \mathbb{R}^n$.*

(b) *There exists $H \geq 0$ such that the algebraic curvature tensor \mathcal{R} defined by (5) satisfies the equation*

$$\text{Tr}(\mathcal{R}_X^2) = H\|X\|^4,$$

for all $X \in \mathbb{R}^n$.

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ and let $X, Y \in \mathbb{R}^n$ be unit, orthonormal vectors. Then (3) is equivalent to

$$\begin{aligned} \sum_{i=1}^n \langle \mathcal{R}(X, e_i)X, \mathcal{R}(X, e_i)Y \rangle &= \sum_{i=1}^n \langle R(X, e_i)X - 2(e_i - \langle X, e_i \rangle X), R(X, e_i)Y + 2\langle Y, e_i \rangle X \rangle \\ &= 2\rho(X, Y) - 2 \sum_{i=1}^n \langle R(X, e_i)Y, e_i - \langle X, e_i \rangle X \rangle - 4 \sum_{i=1}^n \langle e_i - \langle X, e_i \rangle X, \langle Y, e_i \rangle X \rangle = 0, \end{aligned}$$

and so $\sum_{i=1}^n \langle \mathcal{R}(X, e_i)X, \mathcal{R}(X, e_i)Y \rangle = 0$, for any two orthogonal vectors X and Y (not necessarily unit). Consider the function $F : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ defined by $F(X) = \|X\|^{-4} \text{Tr}(\mathcal{R}_X^2) = \|X\|^{-4} \sum_{i=1}^n \langle \mathcal{R}(X, e_i)X, \mathcal{R}(X, e_i)X \rangle$. Then $(\partial_X F)(X) = 0$, as F is 0-homogeneous in X , and $(\partial_Y F)(X) = 0$ for $Y \perp X$, by the equation above. It follows that F is a (non-negative) constant. Conversely, if $\text{Tr}(\mathcal{R}_X^2) = H\|X\|^4$, then the equation $\sum_{i=1}^n \langle \mathcal{R}(X, e_i)X, \mathcal{R}(X, e_i)Y \rangle = 0$ for $X \perp Y$ follows by polarisation. \square

The proof of the Theorem is now concluded by the following purely algebraic fact.

Proposition 2. *Let \mathcal{R} be an algebraic curvature tensor in $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, $n \geq 5$. Suppose that*

(a) *There exists $H \geq 0$ such that for all $X \in \mathbb{R}^n$ we have*

$$\text{Tr}(\mathcal{R}_X^2) = H\|X\|^4. \quad (6)$$

(b) *There exists a direct orthogonal decomposition $\mathbb{R}^n = W_1 \oplus W_2$, with $\dim W_i > 0$, such that*

$$\mathcal{R}(W_1, W_1)W_2 = \mathcal{R}(W_2, W_2)W_1 = 0. \quad (7)$$

Then \mathcal{R} has constant curvature.

To see that Proposition 2 indeed implies the Theorem we note that for the algebraic curvature tensor \mathcal{R} defined by (5), condition (b) is satisfied by (4) if we take $W_1 = E_1$, $W_2 = \oplus_{\lambda=2}^N E_\lambda$, and condition (a), by Lemma 1. Then \mathcal{R} has constant curvature, which by (5) implies that R has constant curvature at p contradicting the fact that the Ricci tensor ρ at p has $N \geq 2$ pairwise distinct eigenvalues.

Proof of Proposition 2. We first give a brief, informal sketch of the proof. Complexifying everything we get that condition (6) is valid for any $X \in \mathbb{C}^n$. Taking a particular $X \in \mathbb{C}^n$ we can symmetrise (6) by the product of two symmetric groups: the permutations of the coordinates of X in W_1 and in W_2 respectively. Then the left-hand side of (6) becomes a quadratic form in the components of \mathcal{R} with coefficients depending on elementary symmetric functions σ_h of the corresponding sets of coordinates (note that the left-hand side of (6) has degree four in the coordinates of X , so we will have only σ_h with $h \leq 4$). We then choose a set of vectors $X^\alpha \in \mathbb{C}^n$ such that $\sum_\alpha \|X^\alpha\|^4 = 0$ and take the sum of the above equations by α . Then the right-hand side of the resulting equation becomes zero, and the left-hand side, if we choose our vectors $X^\alpha \in \mathbb{C}^n$ “in the correct way”, becomes not only real, but a *positive semidefinite* quadratic form in the components of \mathcal{R} . This will give us a set of linear equations in the components of \mathcal{R} which will imply that \mathcal{R} has constant curvature.

Beginning in earnest, we denote $d_1 = \dim W_1 \geq 1$, $d_2 = \dim W_2 \geq 1$, assume that $d_1 \leq d_2$ (recall that $d_1 + d_2 = n \geq 5$), and adopt the following index convention: $1 \leq i, j, k, l \leq d_1$; $d_1 + 1 \leq a, b, c, d \leq d_1 + d_2$. In all summations below, the indices run over the corresponding ranges.

Choose an orthonormal basis $\{e_i, e_a\}$ for $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ such that $W_1 = \text{Span}_i(e_i)$, $W_2 = \text{Span}_a(e_a)$. Then by (7) we have

$$\mathcal{R}_{ijka} = \mathcal{R}_{ijab} = \mathcal{R}_{iabc} = 0, \quad \mathcal{R}_{iajb} = \mathcal{R}_{ibja}, \quad (8)$$

for all the values of the subscripts, where the latter equations follows from the first Bianchi identity.

The algebraic equation (6) holds in the complexification \mathbb{C}^n of \mathbb{R}^n (with the inner product extended from that on \mathbb{R}^n by complex linearity). Let $X = \sum_i x_i e_i + \sum_a y_a e_a$. Then using (8) we get

$$(\mathcal{R}_X)_{ij} = \sum_{kl} x_k x_l \mathcal{R}_{kilj} + \sum_{cd} y_c y_d \mathcal{R}_{cidj}, \quad (\mathcal{R}_X)_{ab} = \sum_{kl} x_k x_l \mathcal{R}_{kalb} + \sum_{cd} y_c y_d \mathcal{R}_{cadb}, \quad (\mathcal{R}_X)_{ia} = \sum_{kc} x_k y_c \mathcal{R}_{cika},$$

and so

$$\begin{aligned} \text{Tr}(\mathcal{R}_X^2) = & \sum_i x_i^4 p_{1|i} + \sum_{i \neq j} x_i^3 x_j p_{2|ij} + \sum_{i \neq j} x_i^2 x_j^2 p_{3|ij} + \sum'_{ijk} x_i^2 x_j x_k p_{4|ijk} + \sum'_{ijkl} x_i x_j x_k x_l p_{5|ijkl} \\ & + \sum_a y_a^4 q_{1|a} + \sum_{a \neq b} y_a^3 y_b q_{2|ab} + \sum_{a \neq b} y_a^2 y_b^2 q_{3|ab} + \sum'_{abc} y_a^2 y_b y_c q_{4|abc} + \sum'_{abcd} y_a y_b y_c y_d q_{5|abcd} \\ & + 2 \sum_{ia} x_i^2 y_a^2 s_{1|ia} + 2 \sum_{a \neq b; i} x_i y_a y_b s_{2|iab} + 2 \sum_{i \neq j; a} x_i x_j y_a s_{3|ija} + 2 \sum_{a \neq b; i \neq j} x_i x_j y_a y_b s_{4|ijab}, \end{aligned} \quad (9)$$

where here and below we denote \sum' the summation by all the pairwise nonequal values of the subscripts in the respective ranges, and where $p_{1|i}, p_{2|ij}, \dots, s_{4|ijab}$ are quadratic forms in the components of \mathcal{R} . In particular,

$$\begin{aligned} p_{1|i} &= \sum_{kl} \mathcal{R}_{ikil}^2 + \sum_{ab} \mathcal{R}_{iaib}^2, \\ p_{3|ij} &= \sum_{kl} \mathcal{R}_{ikil} \mathcal{R}_{jkjl} + \sum_{kl} \mathcal{R}_{ikjl}^2 + \sum_{kl} \mathcal{R}_{ikjl} \mathcal{R}_{jkil} + \sum_{ab} \mathcal{R}_{iaib} \mathcal{R}_{jaib} + 2 \sum_{ab} \mathcal{R}_{iajb}^2, \\ q_{1|a} &= \sum_{cd} \mathcal{R}_{acad}^2 + \sum_{kl} \mathcal{R}_{akal}^2, \\ q_{3|ab} &= \sum_{cd} \mathcal{R}_{acad} \mathcal{R}_{bcdb} + \sum_{cd} \mathcal{R}_{acbd}^2 + \sum_{cd} \mathcal{R}_{acbd} \mathcal{R}_{bcad} + \sum_{kl} \mathcal{R}_{akal} \mathcal{R}_{bkbl} + 2 \sum_{kl} \mathcal{R}_{akbl}^2, \\ s_{1|ia} &= \sum_{kl} \mathcal{R}_{ikil} \mathcal{R}_{akal} + \sum_{cd} \mathcal{R}_{icid} \mathcal{R}_{acad} + \sum_{kc} \mathcal{R}_{akic}^2, \end{aligned}$$

where in the right-most terms of $p_{3|ij}, q_{3|ab}$ and $s_{1|ia}$ we used the last equation of (8). We now take the sum of the expressions for $\text{Tr}(\mathcal{R}_X^2)$ given by (9) by all the permutations of the x_i and of the y_a . The resulting expression $\mathcal{S}(X)$ depends on the elementary symmetric functions $\sigma_h(x), \sigma_h(y)$, $h = 1, 2, 3, 4$, of the variables x_i and y_a respectively (rather than on these variables as such), where we denote $\sigma_1(x) = \sum_i x_i$, $\sigma_2(x) = \sum_{i \neq j} x_i x_j$, $\sigma_3(x) = \sum'_{ijk} x_i x_j x_k$, $\sigma_4(x) = \sum'_{ijkl} x_i x_j x_k x_l$, and similarly, for $\sigma_h(y)$.

Let $\mathcal{Z} \subset \mathbb{C}^n$ be the set of vectors $X = \sum_i x_i e_i + \sum_a y_a e_a$ such that no more than two of the x_i and no more than two of the y_a are nonzero. Assuming that the vector X is chosen in \mathcal{Z} we get $\sigma_3(x) = \sigma_4(x) = \sigma_3(y) = \sigma_4(y) = 0$.

Performing the summation by all the permutations of the x_i and of the y_a in every term on the right-hand side of (9) and then expressing the coefficients in terms of $\sigma_h(x), \sigma_h(y)$ we get

$$\mathcal{S}(X) = \sum_{h=1}^3 A_h(x) P_h + \sum_{h=1}^3 B_h(y) Q_h + \sum_{h=1}^4 C_h(x, y) S_h, \quad (10)$$

with

$$\begin{aligned} A_1(x) &= (d_1 - 1)! d_2! (\sigma_1^4(x) - 4\sigma_1^2(x)\sigma_2(x) + 2\sigma_2^2(x)), \\ A_2(x) &= (d_1 - 2)! d_2! (\sigma_1^2(x)\sigma_2(x) - 2\sigma_2^2(x)), \\ A_3(x) &= (d_1 - 2)! d_2! (2\sigma_2^2(x)), \\ C_1(x, y) &= (d_1 - 1)! (d_2 - 1)! 2(\sigma_1^2(x) - 2\sigma_2(x))(\sigma_1^2(y) - 2\sigma_2(y)), \\ C_2(x, y) &= (d_1 - 1)! (d_2 - 2)! 4(\sigma_1^2(x) - 2\sigma_2(x))\sigma_2(y), \\ C_3(x, y) &= (d_1 - 2)! (d_2 - 1)! 4\sigma_2(x)(\sigma_1^2(y) - 2\sigma_2(y)), \\ C_4(x, y) &= (d_1 - 2)! (d_2 - 2)! 4\sigma_2(x)\sigma_2(y), \end{aligned} \quad (11)$$

with the expressions for $B_h(y)$ obtained from those for $A_h(x)$ by interchanging d_1 and d_2 and replacing x by y , and where we set $\sigma_h(x) = 0$ if $h > d_1$ (respectively, $\sigma_h(y) = 0$ if $h > d_2$) and $m! = 0$ if $m < 0$.

The terms P_h, Q_h and S_h are quadratic forms in the components of \mathcal{R} , where in particular,

$$\begin{aligned}
P_1 &= \sum_i p_{1|i} = \sum_{ikl} \mathcal{R}_{ikil}^2 + \sum_{iab} \mathcal{R}_{iaib}^2, \\
P_3 &= \sum_{i \neq j} p_{3|ij} = \sum_{i \neq j} \left(\sum_{kl} \mathcal{R}_{ikil} \mathcal{R}_{jkjl} + \sum_{kl} \mathcal{R}_{ikjl}^2 + \sum_{kl} \mathcal{R}_{ikjl} \mathcal{R}_{jkil} + \sum_{ab} \mathcal{R}_{iaib} \mathcal{R}_{jaib} + 2 \sum_{ab} \mathcal{R}_{iajb}^2 \right), \\
Q_1 &= \sum_a q_{1|a} = \sum_{acd} \mathcal{R}_{acad}^2 + \sum_{kla} \mathcal{R}_{akal}^2, \\
Q_3 &= \sum_{a \neq b} q_{3|ab} = \sum_{a \neq b} \left(\sum_{cd} \mathcal{R}_{acad} \mathcal{R}_{bcbd} + \sum_{cd} \mathcal{R}_{acbd}^2 + \sum_{cd} \mathcal{R}_{acbd} \mathcal{R}_{bcad} + \sum_{kl} \mathcal{R}_{akal} \mathcal{R}_{bkbl} + 2 \sum_{kl} \mathcal{R}_{akbl}^2 \right), \\
S_1 &= \sum_{ia} s_{1|ia} = \sum_{ikla} \mathcal{R}_{ikil} \mathcal{R}_{akal} + \sum_{iacd} \mathcal{R}_{icid} \mathcal{R}_{acad} + \sum_{ikac} \mathcal{R}_{akic}^2.
\end{aligned} \tag{12}$$

From (6) by (11) we get

$$\begin{aligned}
\mathcal{S}(X) &= H d_1! d_2! \|X\|^4 = H d_1! d_2! (\sigma_1^2(x) - 2\sigma_2(x) + \sigma_1^2(y) - 2\sigma_2(y))^2 \\
&= H(d_1 A_1(x) + d_1(d_1 - 1) A_3(x) + d_2 B_1(y) + d_2(d_2 - 1) B_3(y) + d_1 d_2 C_1(x, y)).
\end{aligned} \tag{13}$$

We consider two cases.

Case 1. $d_1 = 1$. Take $X = e_1 + i e_2$, so that $\sigma_1(x) = 1, \sigma_1(y) = i, \sigma_2(x) = \sigma_2(y) = 0$. Then from (11) we get $A_1(x) = d_2!, B_1(y) = (d_2 - 1)!, C_1(x, y) = -2(d_2 - 1)!$, and all the other $A_h(x), B_h(y), C_h(x, y)$ are zeros. It follows from (13) that $\mathcal{S}(X) = 0$, and from (10), that $\mathcal{S}(X) = (d_2 - 1)!(d_2 P_1 + Q_1 - 2S_1)$, and so $d_2 P_1 + Q_1 - 2S_1 = 0$. But from (12)

$$\begin{aligned}
d_2 P_1 + Q_1 - 2S_1 &= d_2 \sum_{ab} \mathcal{R}_{1a1b}^2 + \sum_{acd} \mathcal{R}_{acad}^2 + \sum_a \mathcal{R}_{a1a1}^2 - 2 \sum_{acd} \mathcal{R}_{1c1d} \mathcal{R}_{acad} - 2 \sum_{ac} \mathcal{R}_{1a1c}^2 \\
&= \sum_{a \neq c, d} (\mathcal{R}_{1c1d} - \mathcal{R}_{acad})^2.
\end{aligned}$$

Then $\mathcal{R}_{1c1d} = \mathcal{R}_{acad}$, for all a, c, d such that $a \neq c, d$. As the choice of the orthonormal basis $\{e_a\}$ for W_2 is arbitrary and as $\mathcal{R}_{1acd} = 0$ by (8), the claim easily follows.

Case 2. $d_1 \geq 2$. The proof is similar to that in Case 1, but we need more than one vector X .

Let $\mathcal{X} = \{X^1, \dots, X^m\}$ be a set of vectors $X^\alpha = \sum_i x_i^\alpha e_i + \sum_a y_a^\alpha e_a \in \mathcal{Z}$. Denote $\mathcal{S}(\mathcal{X}) = \sum_{\alpha=1}^m \mathcal{S}(X^\alpha)$ and $A_h(\mathcal{X}) = \sum_{\alpha=1}^m A_h(x^\alpha)$, $B_h(\mathcal{X}) = \sum_{\alpha=1}^m B_h(y^\alpha)$, $C_h(\mathcal{X}) = \sum_{\alpha=1}^m C_h(x^\alpha, y^\alpha)$. Then by (10)

$$\mathcal{S}(\mathcal{X}) = \sum_{h=1}^3 A_h(\mathcal{X}) P_h + \sum_{h=1}^3 B_h(\mathcal{X}) Q_h + \sum_{h=1}^4 C_h(\mathcal{X}) S_h, \tag{14}$$

and by (13),

$$\mathcal{S}(\mathcal{X}) = H(d_1 A_1(\mathcal{X}) + d_1(d_1 - 1) A_3(\mathcal{X}) + d_2 B_1(\mathcal{X}) + d_2(d_2 - 1) B_3(\mathcal{X}) + d_1 d_2 C_1(\mathcal{X})). \tag{15}$$

We require the following fact.

Lemma 2. *For arbitrary complex numbers a_h, b_h , $h = 1, 2, 3$, and c_h , $h = 1, 2, 3, 4$, there exists a set $\mathcal{X} = \{X^1, \dots, X^m\} \subset \mathcal{Z}$ such that $A_h(\mathcal{X}) = a_h$, $B_h(\mathcal{X}) = b_h$, $h = 1, 2, 3$, and $C_h(\mathcal{X}) = c_h$, $h = 1, 2, 3, 4$.*

Proof. The set of 10-dimensional vectors $(A_1(\mathcal{X}), A_2(\mathcal{X}), A_3(\mathcal{X}), B_1(\mathcal{X}), B_2(\mathcal{X}), B_3(\mathcal{X}), C_1(\mathcal{X}), C_2(\mathcal{X}), C_3(\mathcal{X}), C_4(\mathcal{X}))$ is a linear subspace of \mathbb{C}^{10} . Suppose that subspace is proper. Then there exists a nontrivial linear combination $\sum_{h=1}^3 \mu_h A_h(\mathcal{X}) + \sum_{h=1}^3 \nu_h B_h(\mathcal{X}) + \sum_{h=1}^4 \lambda_h C_h(\mathcal{X})$ which vanishes for any \mathcal{X} , and so $\sum_{h=1}^3 \mu_h A_h(x) + \sum_{h=1}^3 \nu_h B_h(y) + \sum_{h=1}^4 \lambda_h C_h(x, y) = 0$ for any $X \in \mathcal{Z}$. But from (11), taking $y = 0, \sigma_1(x) = 1, \sigma_2(x) = 0$ we get $\mu_1 = 0$, then taking $y = 0, \sigma_1(x) = 1, \sigma_2(x) = \sqrt{2}$ we get $\mu_3 = 0$, and then taking $y = 0, \sigma_1(x) = 0, \sigma_2(x) = 1$ we get $\mu_2 = 0$. Similarly $\nu_1 = \nu_2 = \nu_3 = 0$. Then again from (11) taking $\sigma_1(x) = \sigma_1(y) = 1, \sigma_2(x) = \sigma_2(y) = 0$ we get $\lambda_1 = 0$, then taking $\sigma_1(x) = 1, \sigma_2(x) = 0, \sigma_2(y) = 1$ we get $\lambda_2 = 0$, then taking $\sigma_1(y) = 1, \sigma_2(y) = 0, \sigma_2(x) = 1$ we get $\lambda_3 = 0$, and then taking $\sigma_2(x) = \sigma_2(y) = 1$ we get $\lambda_4 = 0$. \square

Take positive real numbers ξ and η satisfying the following inequalities:

$$\begin{aligned} \mu &:= (d_2 - 1)\eta + (d_2 - d_1 - 1)\xi > 0, & \nu &:= (d_1 - 1)\xi + (d_1 - d_2 - 1)\eta > 0, \\ (d_1 - d_2 - 2)\eta + d_1\xi &\geq 0, & (d_2 - d_1 - 2)\xi + d_2\eta &\geq 0 \end{aligned} \quad (16)$$

(recall that $2 \leq d_1 \leq d_2$ and $d_1 + d_2 > 4$). If $d_2 = d_1$ we can take $\xi = \eta = 1$; if $d_2 \geq d_1 + 2$ we can take $\eta = 1$ and $\xi > \max((d_1 - 1)^{-1}(d_2 - d_1 + 1), d_1^{-1}(d_2 - d_1 + 2))$; if $d_2 \geq d_1 + 1$ we can take $\eta = 1$ and $\xi \in (\max(2(d_1 - 1)^{-1}, 3d_1^{-1}), d_1 + 1)$.

By Lemma 2 we can choose \mathcal{X} in such a way that

$$\begin{aligned} A_2(\mathcal{X}) &= B_2(\mathcal{X}) = C_2(\mathcal{X}) = C_3(\mathcal{X}) = C_4(\mathcal{X}) = 0, & A_3(\mathcal{X}) &= \xi, & B_3(\mathcal{X}) &= \eta, \\ C_1(\mathcal{X}) &= -2(\xi + \eta), & A_1(\mathcal{X}) &= (d_2 - d_1 + 1)\xi + d_2\eta, & B_1(\mathcal{X}) &= (d_1 - d_2 + 1)\eta + d_1\xi. \end{aligned}$$

Then by (15) $\mathcal{S}(\mathcal{X}) = 0$, and so from (14) and (12) we get

$$\begin{aligned} 0 &= A_1(\mathcal{X})P_1 + A_3(\mathcal{X})P_3 + B_1(\mathcal{X})Q_1 + B_3(\mathcal{X})Q_3 + C_1(\mathcal{X})S_1 \\ &= ((d_2 - d_1 + 1)\xi + d_2\eta)P_1 + \xi P_3 + ((d_1 - d_2 + 1)\eta + d_1\xi)Q_1 + \eta Q_3 - 2(\xi + \eta)S_1 \\ &= ((d_2 - d_1 + 1)\xi + d_2\eta) \left(\sum_{ikl} \mathcal{R}_{ikil}^2 + \sum_{iab} \mathcal{R}_{iaib}^2 \right) \\ &\quad + \xi \sum_{i \neq j} \left(\sum_{kl} \mathcal{R}_{ikil} \mathcal{R}_{jkjl} + \sum_{kl} \mathcal{R}_{ikjl}^2 + \sum_{kl} \mathcal{R}_{ikjl} \mathcal{R}_{jkil} + \sum_{ab} \mathcal{R}_{iaib} \mathcal{R}_{jaib} + 2 \sum_{ab} \mathcal{R}_{iajb}^2 \right) \\ &\quad + ((d_1 - d_2 + 1)\eta + d_1\xi) \left(\sum_{acd} \mathcal{R}_{acad}^2 + \sum_{kla} \mathcal{R}_{akal}^2 \right) \\ &\quad + \eta \sum_{a \neq b} \left(\sum_{cd} \mathcal{R}_{acad} \mathcal{R}_{bcbd} + \sum_{cd} \mathcal{R}_{acbd}^2 + \sum_{cd} \mathcal{R}_{acbd} \mathcal{R}_{bcad} + \sum_{kl} \mathcal{R}_{akal} \mathcal{R}_{bkbl} + 2 \sum_{kl} \mathcal{R}_{akbl}^2 \right) \\ &\quad - 2(\xi + \eta) \left(\sum_{ikla} \mathcal{R}_{ikil} \mathcal{R}_{akal} + \sum_{iacd} \mathcal{R}_{icid} \mathcal{R}_{acad} + \sum_{ikac} \mathcal{R}_{akic}^2 \right) \end{aligned} \quad (17)$$

We now show that the quadratic form \mathcal{Q} on the right-hand side of (17) is positive semidefinite in the components of \mathcal{R} . The form \mathcal{Q} does not contain the components \mathcal{R}_{iajb} with $i \neq j$, $a \neq b$, as the corresponding terms cancel out. Furthermore, we have $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 + \mathcal{Q}_4$, where \mathcal{Q}_1 only involves the components \mathcal{R}_{ikjl} with i, j, k, l pairwise non-equal; \mathcal{Q}_2 , only the components \mathcal{R}_{abcd} with a, b, c, d pairwise non-equal; \mathcal{Q}_3 , only the components $\mathcal{R}_{ijij}, \mathcal{R}_{abab}, \mathcal{R}_{iaia}$; and \mathcal{Q}_4 , only the components $\mathcal{R}_{ijik}, \mathcal{R}_{ibic}, \mathcal{R}_{ajak}, \mathcal{R}_{abac}$ with $j \neq k$ and $b \neq c$.

We have

$$\mathcal{Q}_1 = \frac{1}{2}\xi \sum'_{ijkl} (\mathcal{R}_{ikjl} + \mathcal{R}_{jkil})^2, \quad \mathcal{Q}_2 = \frac{1}{2}\eta \sum'_{acbd} (\mathcal{R}_{acbd} + \mathcal{R}_{adbc})^2.$$

To simplify the form \mathcal{Q}_3 we denote $U_k = \sum_i \mathcal{R}_{ikik}, V_k = \sum_a \mathcal{R}_{akak}, U_a = \sum_c \mathcal{R}_{caca}, V_a = \sum_i \mathcal{R}_{iaia}$. Then

$$\begin{aligned} \mathcal{Q}_3 &= ((d_2 - d_1 + 1)\xi + d_2\eta) \sum_{ik} \mathcal{R}_{ikik}^2 + \xi \sum_{ijk} \mathcal{R}_{ikik} \mathcal{R}_{jkjk} + ((d_1 - d_2 + 1)\eta + d_1\xi) \sum_{ac} \mathcal{R}_{acac}^2 + \eta \sum_{abc} \mathcal{R}_{acac} \mathcal{R}_{bcbc} \\ &\quad + ((d_2 - 2)\xi + (d_1 - 2)\eta) \sum_{ia} \mathcal{R}_{iaia}^2 + \xi \sum_{ija} \mathcal{R}_{iaia} \mathcal{R}_{jaja} + \eta \sum_{abk} \mathcal{R}_{akak} \mathcal{R}_{bkbk} \\ &\quad - 2(\xi + \eta) \left(\sum_{ika} \mathcal{R}_{ikik} \mathcal{R}_{akak} + \sum_{iac} \mathcal{R}_{icic} \mathcal{R}_{acac} \right) \\ &= ((d_2 - d_1 + 1)\xi + d_2\eta) \sum_{ik} \mathcal{R}_{ikik}^2 + \xi \sum_k U_k^2 + ((d_1 - d_2 + 1)\eta + d_1\xi) \sum_{ac} \mathcal{R}_{acac}^2 + \eta \sum_c U_c^2 \\ &\quad + ((d_2 - 2)\xi + (d_1 - 2)\eta) \sum_{ia} \mathcal{R}_{iaia}^2 + \xi \sum_a V_a^2 + \eta \sum_k V_k^2 - 2(\xi + \eta) \left(\sum_k U_k V_k + \sum_c U_c V_c \right). \end{aligned}$$

Using the fact that $\sum_{ik} \mathcal{R}_{ikik}^2 = \frac{1}{2(d_1-1)} \sum_{kij} (\mathcal{R}_{ikik} - \mathcal{R}_{jkjk})^2 + \frac{1}{d_1-1} \sum_k U_k^2$, $\sum_{ac} \mathcal{R}_{acac}^2 = \frac{1}{2(d_2-1)} \sum_{acd} (\mathcal{R}_{acac} - \mathcal{R}_{adad})^2 + \frac{1}{d_2-1} \sum_a U_a^2$, and $((d_2 - 2)\xi + (d_1 - 2)\eta) \sum_{ia} \mathcal{R}_{iaia}^2 = (\mu + \nu) \sum_{ia} \mathcal{R}_{iaia}^2 = \mu \left(\frac{1}{2d_1} \sum_{aij} (\mathcal{R}_{iaia} - \mathcal{R}_{jaja})^2 + \right.$

$\frac{1}{d_1} \sum_a V_a^2) + \nu(\frac{1}{2d_2} \sum_{abi} (\mathcal{R}_{iaia} - \mathcal{R}_{ibib})^2 + \frac{1}{d_2} \sum_i V_i^2)$, where μ and ν are given by (16), we obtain

$$\begin{aligned} \mathcal{Q}_3 = & \frac{(d_2 - d_1 + 1)\xi + d_2\eta}{2(d_1 - 1)} \sum_{kij} (\mathcal{R}_{ikik} - \mathcal{R}_{jkjk})^2 + \frac{(d_1 - d_2 + 1)\eta + d_1\xi}{2(d_2 - 1)} \sum_{acd} (\mathcal{R}_{acac} - \mathcal{R}_{adad})^2 \\ & + \frac{\mu}{2d_1} \sum_{aij} (\mathcal{R}_{iaia} - \mathcal{R}_{jaja})^2 + \frac{\nu}{2d_2} \sum_{abi} (\mathcal{R}_{iaia} - \mathcal{R}_{ibib})^2 \\ & + (\xi + \eta)d_2(d_1 - 1) \sum_k \left(\frac{1}{d_1 - 1} U_k - \frac{1}{d_2} V_k \right)^2 + (\xi + \eta)d_1(d_2 - 1) \sum_a \left(\frac{1}{d_2 - 1} U_a - \frac{1}{d_1} V_a \right)^2. \end{aligned} \quad (18)$$

We now consider the quadratic form \mathcal{Q}_4 . Collecting the terms we obtain

$$\begin{aligned} \mathcal{Q}_4 = & \sum_{k \neq l} \left[((d_2 - d_1 + 4)\xi + d_2\eta) \sum_i \mathcal{R}_{ikil}^2 + ((d_1 - d_2 - 2)\eta + d_1\xi) \sum_a \mathcal{R}_{akal}^2 \right. \\ & \left. + \xi \sum_{ij} \mathcal{R}_{ikil} \mathcal{R}_{jkjl} + \eta \sum_{ab} \mathcal{R}_{akal} \mathcal{R}_{bkbl} - 2(\xi + \eta) \sum_{ia} \mathcal{R}_{ikil} \mathcal{R}_{akal} \right] \\ & + \sum_{c \neq d} \left[((d_1 - d_2 + 4)\eta + d_1\xi) \sum_a \mathcal{R}_{acac}^2 + ((d_2 - d_1 - 2)\xi + d_2\eta) \sum_i \mathcal{R}_{icid}^2 \right. \\ & \left. + \eta \sum_{ab} \mathcal{R}_{acac} \mathcal{R}_{bcdb} + \xi \sum_{ij} \mathcal{R}_{icid} \mathcal{R}_{jcjd} - 2(\xi + \eta) \sum_{c \neq d; ia} \mathcal{R}_{icid} \mathcal{R}_{acac} \right]. \end{aligned}$$

The form \mathcal{Q}_4 is positive semidefinite if the expressions in the square brackets are non-negative. For the first one, it suffices to show that the quadratic form $f = ((d_2 - d_1 + 4)\xi + d_2\eta) \sum_i u_i^2 + ((d_1 - d_2 - 2)\eta + d_1\xi) \sum_a v_a^2 + \xi \sum_{ij} u_i u_j + \eta \sum_{ab} v_a v_b - 2(\xi + \eta) \sum_{ia} u_i v_a$ in the variables $u_1, \dots, u_{d_1-2}, v_1, \dots, v_{d_2}$ is positive semidefinite. Note that $\xi, \eta > 0$ and that $(d_2 - d_1 + 4)\xi + d_2\eta, (d_1 - d_2 - 2)\eta + d_1\xi \geq 0$ by (16). If $d_1 = 2$, there is no u_i 's and $f \geq 0$ trivially. If $d_1 > 2$, then by an orthogonal change of variables $\{u_i\} \mapsto \{u'_i\}$, $\{v_a\} \mapsto \{v'_a\}$ such that $u'_1 = (d_1 - 2)^{-1/2} \sum_i u_i$, $v'_1 = d_2^{-1/2} \sum_a v_a$, we get $f = ((d_2 + 2)\xi + d_2\eta) u_1'^2 + ((d_1 - 2)\eta + d_1\xi) v_1'^2 - 2(\xi + \eta) \sqrt{d_2(d_1 - 2)} u_1' v_1' + ((d_2 - d_1 + 4)\xi + d_2\eta) \sum_{i>1} u_i'^2 + ((d_1 - d_2 - 2)\eta + d_1\xi) \sum_{a>1} v_a'^2$, and so $f \geq 0$, as $((d_2 + 2)\xi + d_2\eta)((d_1 - 2)\eta + d_1\xi) - d_2(d_1 - 2)(\xi + \eta)^2 = 2(d_1 + d_2)\xi^2 + 2(d_1 + d_2 - 2)\xi\eta > 0$. A similar argument for the second square bracket shows that $\mathcal{Q}_4 \geq 0$.

It now follows from (17) that $\mathcal{Q}_1 = \mathcal{Q}_2 = \mathcal{Q}_3 = 0$, which by (18) implies that $\mathcal{R}_{ikik} = \mathcal{R}_{jkjk}, \mathcal{R}_{acac} = \mathcal{R}_{adad}, \mathcal{R}_{iaia} = \mathcal{R}_{jaja}, \mathcal{R}_{iaia} = \mathcal{R}_{ibib}$, and $d_2 U_k = (d_1 - 1) V_k, d_1 U_a = (d_2 - 1) V_a$. As the choice of the bases for W_1, W_2 is arbitrary, the claim follows. \square

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